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Stability and vibration of thick laminated composite sector plates

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Abstract

This study presents a simple analytical formulation for the eigenvalue problem of buckling and free vibration analysis of shear deformable laminated sector plates made up of cylindrically orthotropic layers. The non-axisymmetric formulation in cylindrical coordinates is discretized in space domain in terms of two-dimensional Chebyshev polynomials. Several combinations of simply supported, clamped and free edge conditions are considered. Convergence study has been carried out and the obtained results are compared with the results of laminated square plates and isotropic sector plates available in literature. Extensive results pertaining to critical buckling loads and natural frequencies are presented. Effects of boundary conditions, number of layers, moduli ratio, rotary and in-plane inertia, plate thickness, sector angle and annularity are studied.

1. Introduction

Mechanics of plates is an established research field. Many papers giving buckling and vibration analysis of plates of various shapes and subjected to various combinations of edge conditions have been published.

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Nomenclature

A_{ij}, B_{ij}, D_{ij} extensional stiffnesses, bending–extension coupling stiffnesses ($i,j = 1,2,6$) and bending stiffnesses of the laminated plate
A_{ij}, B_{ij}, D_{ij} ($i,j = 4,5$) transverse shear stiffnesses of the laminated plate
$E_r^{(k)}, E_\theta^{(k)}$ elastic moduli in respective directions for the kth layer
$G_{zr}^{(k)}, G_{r\theta}^{(k)}, G_{\theta z}^{(k)}$ moduli of rigidity in respective planes for the kth layer
I_0, I_1, I_2 linear inertia, linear–rotary coupling inertia and rotary inertia
$M_r, M_\theta, M_{r\theta}$ moment resultants
$N_r, N_\theta, N_{r\theta}$ in-plane force resultants
N_{ip} isotropic in-plane compressive load
N_{cr} critical buckling load
$Q_{ij}^{(k)}$ ($i,j = 1,2,6$) plane stress reduced stiffnesses for the kth layer
$Q_{ij}^{(k)}$ ($i,j = 4,5$) transverse shear reduced stiffnesses for the kth layer
Q_r, Q_θ transverse shear stress resultants
k^2 shear correction factor ($\frac{5}{6}$)
n number of layers in the laminated plate
r_i, r_o, h internal and external radii and total thickness of the laminated sector plate
r^*, θ^* radial (m) and angular (rad) coordinates
$r = (r^* - r_i)/(r_o - r_i)$ non-dimensional radial coordinate
t time (s)
u, v, w linear displacement components (m) in r, θ, z directions
u_0, v_0, w_0 linear displacement components (m) of a point on the plane $z = 0$ in r, θ, z directions
u_{ij}, v_{ij}, w_{ij} coefficients in Chebyshev series expansions of non-dimensionalized linear displacement components u_0, v_0, w_0
α, λ annularity (r_i/r_o), thickness ratio (h/r_o)
$\varepsilon_r^{(0)}, \varepsilon_\theta^{(0)}, \varepsilon_{r\theta}^{(0)}$ in-plane strain components at the $z = 0$ plane
$\kappa_r^{(0)}, \kappa_\theta^{(0)}, \kappa_{r\theta}^{(0)}$ curvatures of the $z = 0$ plane
$\theta = (\theta^*/\Theta)$ non-dimensional angular coordinate
ϕ_r, ϕ_θ angular displacements in rz and θz planes
$\phi_{r,ij}, \phi_{\theta,ij}$ coefficients in Chebyshev series expansions of angular displacement components ϕ_r, ϕ_θ
$\nu_{r\theta}^{(k)}$ a Poisson's ratio in $r\theta$ plane for the kth layer
$\rho^{(k)}$ mass density for the kth layer
ω a natural frequency of vibration
Θ total included sector angle (rad)

Buckling analyses of both isotropic as well as non-isotropic and non-homogeneous rectangular plates have been carried out extensively. Both linear formulations involving different kinds of complicating effects [1,2] as well as nonlinear [3–5] studies are being reported.

Nath and Kumar [6,7] have presented nonlinear analysis of laminated axisymmetric shells.

Free vibration problems of isotropic annular sector plates have been solved earlier by using the exact method applied to thin plate model [8], and, finite element method [9,10], the finite strip method [11,12] and the Rayleigh–Ritz method applied to thin plate model [13] and Mindlin plate model [14,15]. Liew et al. [16] gave three-dimensional analysis for vibration of isotropic annular sector plates. Srinivasan and Thiruvenkatachari [17] applied the boundary element method to solve this problem for transversely isotropic Mindlin sector plate. Liew et al. [18] have given a survey of research on vibration of thick plates of various shapes.

Rubin [19] studied the stability of orthotropic sector plates. Harik [20] presented the buckling analysis of sector plates with clamped radial edges. Both these works used Kirchhoff plate theory. Wang and Xiang [21] proposed a method for deducing buckling loads of isotropic sectorial Mindlin plates from the Kirchhoff plate results. They gave an exact relationship between buckling loads of Mindlin model and of Kirchhoff model applied to complete sector plates having simply supported radial and circumferential edges. The relationship was further shown to be applicable to clamped or free circumferential edges with the help of modifying factors obtained by regression analysis.

McGee et al. [22] presented exact solutions for free vibration analysis of isotropic sector plates having simply supported radial edges and all possible combinations of free, simply supported and clamped boundary conditions on circular edges. Liew and Liu [23] and Liu and Liew [24] presented differential quadrature method for vibration analysis of shear deformable isotropic sector plates.

It is clear from literature that there is a need for an analytical solution that handles several kinds of boundary conditions as well as non-isotropy and non-homogeneity of the material in the free vibration and buckling analysis of sector plates. The present study is an effort in this direction.

Considering the first-order shear deformation theory, the vibration and stability analyses of moderately thick laminated sector plates made up of cylindrically orthotropic layers are carried out analytically using Chebyshev polynomials. The advantage of expressing the field variables in terms of Chebyshev polynomials lies in the fact that a polynomial of a particular degree with leading coefficient unity approximating zero in the minimax sense is some multiple of the Chebyshev polynomial of that degree [25]. Thus the error in the field variables as also in the field equations automatically gets minimized (zeroed) in the minimax sense.

The eigenvalue problem is formulated by discretizing the governing equilibrium equations and boundary conditions with the help of two-dimensional Chebyshev polynomials [26] and then solved using the procedure given by Wilkinson and Reinsch [27] for non-symmetric matrices. The formulation enables determination of mode shapes also.

The influence of boundary conditions, orthotropy, number of layers, thickness ratio λ, annularity α, sector angle Θ, rotary inertia and inplane inertia on the frequency and critical buckling loads are investigated.

2. Mathematical formulation

Fig. 1 shows the geometry of the sector plate. Each lamina is considered to be cylindrically orthotropic with the fiber orientation being either in the radial or circumferential direction. Considering first-order shear deformation theory, the displacement fields are expressed as follows:

$$\begin{aligned} u(r,\theta,z,t) &= u_0(r,\theta,t) + z\phi_r(r,\theta,t) \\ v(r,\theta,z,t) &= v_0(r,\theta,t) + z\phi_\theta(r,\theta,t) \\ w(r,\theta,z,t) &= w_0(r,\theta,t) \end{aligned} \tag{1}$$

Strain displacement equations can be expressed as follows. For simplicity the suffix '0' is dropped from the linear displacement components (u, v and w) and these are taken to imply displacements of the mid-plane.

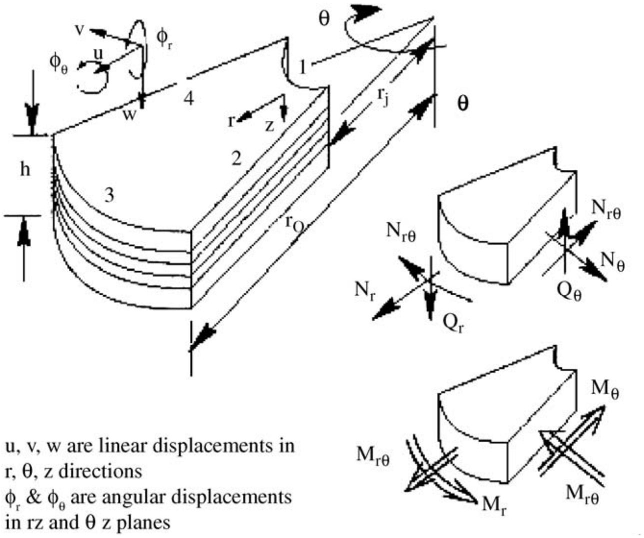

Fig. 1. Geometry of the problem.

In-plane strains for the mid-plane are

$$\varepsilon_r^{(0)} = \frac{1}{r_o(1-\alpha)}\frac{\partial u}{\partial r}, \tag{2}$$

$$\varepsilon_\theta^{(0)} = \left(\frac{1}{\{r(1-\alpha)+\alpha\}r_o}\right)\left(u + \frac{1}{\Theta}\frac{\partial v}{\partial \theta}\right), \tag{3}$$

$$\varepsilon_{r\theta}^{(0)} = \left(\frac{1}{\{r(1-\alpha)+\alpha\}r_o}\right)\left(\frac{1}{\Theta}\frac{\partial u}{\partial \theta}\right) + \frac{1}{r_o(1-\alpha)}\frac{\partial v}{\partial r} - \frac{v}{\{r(1-\alpha)+\alpha\}r_o}. \tag{4}$$

The curvatures are

$$\kappa_r^{(0)} = \frac{1}{r_o(1-\alpha)}\frac{\partial \phi_r}{\partial r}, \tag{5}$$

$$\kappa_\theta^{(0)} = \left(\frac{1}{\{r(1-\alpha)+\alpha\}r_o}\right)\left(\phi_r + \frac{1}{\Theta}\frac{\partial \phi_\theta}{\partial \theta}\right), \tag{6}$$

$$\kappa_{r\theta}^{(0)} = \left(\frac{1}{\{r(1-\alpha)+\alpha\}r_o}\right)\left(\frac{1}{\Theta}\frac{\partial \phi_r}{\partial \theta}\right) + \frac{1}{r_o(1-\alpha)}\frac{\partial \phi_\theta}{\partial r} - \frac{\phi_\theta}{\{r(1-\alpha)+\alpha\}r_o}. \tag{7}$$

The shear strains in rz and θz planes are

$$e_{rz} = \phi_r + \frac{1}{r_o(1-\alpha)}\frac{\partial w}{\partial r}, \tag{8}$$

$$e_{\theta z} = \phi_\theta + \left(\frac{1}{\{r(1-\alpha)+\alpha\}r_o}\right)\left(\frac{1}{\Theta}\frac{\partial w}{\partial \theta}\right). \tag{9}$$

The constitutive equations are thus obtained as

$$\begin{Bmatrix} N_r \\ N_\theta \\ N_{r\theta} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_r^{(0)} \\ \varepsilon_\theta^{(0)} \\ \varepsilon_{r\theta}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_r^{(0)} \\ \kappa_\theta^{(0)} \\ \kappa_{r\theta}^{(0)} \end{Bmatrix}, \tag{10}$$

$$\begin{Bmatrix} M_r \\ M_\theta \\ M_{r\theta} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_r^{(0)} \\ \varepsilon_\theta^{(0)} \\ \varepsilon_{r\theta}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_r^{(0)} \\ \kappa_\theta^{(0)} \\ \kappa_{r\theta}^{(0)} \end{Bmatrix}, \tag{11}$$

$$\begin{Bmatrix} Q_\theta \\ Q_r \end{Bmatrix} = k^2 \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} e_{\theta z} \\ e_{zr} \end{Bmatrix}, \tag{12}$$

where

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n}(z_{k+1}-z_k)Q_{ij}^{(k)}, \\ B_{ij} &= \frac{1}{2}\sum_{k=1}^{n}(z_{k+1}^2-z_k^2)Q_{ij}^{(k)}, \\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{n}(z_{k+1}^3-z_k^3)Q_{ij}^{(k)}, \quad i,j = 1,2,6, \end{aligned} \tag{13}$$

$$A_{ij} = \sum_{k=1}^{n}(z_{k+1}-z_k)Q_{ij}^{(k)}, \quad i,j = 4,5. \tag{14}$$

The stiffness constants $Q_{ij}^{(k)}$ for the kth layer are

$$\begin{aligned} Q_{11}^{(k)} &= \frac{E_r^{(k)}}{[1-v_{r\theta}^2(E_\theta/E_r)]}, \\ Q_{22}^{(k)} &= \frac{E_\theta^{(k)}}{[1-v_{r\theta}^2(E_\theta/E_r)]}, \\ Q_{12}^{(k)} &= \frac{v_{r\theta}E_\theta^{(k)}}{[1-v_{r\theta}^2(E_\theta/E_r)]}, \end{aligned} \tag{15}$$

$$Q_{66} = G_{r\theta},$$
$$Q_{16} = Q_{26} = 0, \tag{16}$$

$$Q_{44} = G_{\theta z}, Q_{55} = G_{zr}, Q_{45} = 0. \tag{17}$$

For the cases of symmetric and unsymmetric laminated plates made up of cylindrically orthotropic layers having fiber orientations either in radial or circumferential directions, we have

$$A_{ij} = B_{ij} = D_{ij} = 0 \quad \text{for } i = 1, 2 \text{ and } j = 6.$$

And also,

$$A_{45} = 0.$$

Including rotary inertia, the equations of motion in terms of stress resultants and non-dimensional coordinates are obtained as

$$\frac{1}{r_o(1-\alpha)}\frac{\partial N_r}{\partial r} + \frac{1}{\{r(1-\alpha)+\alpha\}r_o}\frac{\partial N_{r\theta}}{\partial \theta} + \frac{N_r - N_\theta}{\{r(1-\alpha)+\alpha\}r_o} = I_0\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + I_1\frac{\mathrm{d}^2 \phi_r}{\mathrm{d}t^2}, \tag{18}$$

$$\frac{1}{r_o(1-\alpha)}\frac{\partial N_{r\theta}}{\partial r} + \frac{1}{\{r(1-\alpha)+\alpha\}r_o}\frac{\partial N_\theta}{\partial \theta} + \frac{2N_{r\theta}}{\{r(1-\alpha)+\alpha\}r_o} = I_0\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + I_1\frac{\mathrm{d}^2 \phi_\theta}{\mathrm{d}t^2}, \tag{19}$$

$$\begin{aligned}&\frac{1}{r_o(1-\alpha)}\frac{\partial Q_r}{\partial r} + \frac{1}{\{r(1-\alpha)+\alpha\}r_o}\frac{\partial Q_\theta}{\partial \theta} + \frac{Q_r}{\{r(1-\alpha)+\alpha\}r_o} \\ &\quad + N_{\mathrm{ip}}\left[\left(\frac{1}{r_o^2(1-\alpha)^2}\frac{\partial^2 w}{\partial r^2}\right) + \frac{1}{\{r(1-\alpha)+\alpha\}r_o}\left(\frac{1}{r_o(1-\alpha)}\frac{\partial w}{\partial r}\right)\right. \\ &\quad \left. + \frac{1}{\Theta^2\{r(1-\alpha)+\alpha\}^2 r_o^2}\frac{\partial^2 w}{\partial \theta^2}\right] = I_0\frac{\mathrm{d}^2 w}{\mathrm{d}t^2},\end{aligned} \tag{20}$$

$$\frac{1}{r_o(1-\alpha)}\frac{\partial M_r}{\partial r} + \frac{1}{\{r(1-\alpha)+\alpha\}r_o}\frac{\partial M_{r\theta}}{\partial \theta} + \frac{M_r - M_\theta}{\{r(1-\alpha)+\alpha\}r_o} - Q_r = I_2\frac{\mathrm{d}^2 \phi_r}{\mathrm{d}t^2} + I_1\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}, \tag{21}$$

$$\frac{1}{r_o(1-\alpha)}\frac{\partial M_{r\theta}}{\partial r} + \frac{1}{\{r(1-\alpha)+\alpha\}r_o}\frac{\partial M_\theta}{\partial \theta} + \frac{2M_{r\theta}}{\{r(1-\alpha)+\alpha\}r_o} - Q_\theta = I_2\frac{\mathrm{d}^2 \phi_\theta}{\mathrm{d}t^2} + I_1\frac{\mathrm{d}^2 v}{\mathrm{d}t^2}, \tag{22}$$

where the inertias are defined as

$$(I_0, I_1, I_2) = \sum_{k=1}^{n}\int_{z_k}^{z_{k+1}} \rho^{(k)}(1, z, z^2)\,\mathrm{d}z. \tag{23}$$

On substituting the constitutive equations (10)–(12) and thee strain–displacement equations (2)–(9), displacement equations of equilibrium are obtained and are given in Appendix A.

3. Method of solution

A general function $\Psi(r,\theta)$ can be approximated in space domain by a finite degree double Chebyshev polynomial as

$$\Psi(r,\theta) = \sum_{i=0}^{M} \sum_{j=0}^{N} \delta_{ij} \Psi_{ij} T_i(r) T_j(\theta), \tag{24}$$

where

$$\delta_{ij} = \begin{cases} \frac{1}{2} & \text{for } (i=0 \text{ and } j>0) \text{ or } (i>0 \text{ and } j=0), \\ \frac{1}{4} & \text{for } i=j=0, \\ 1 & \text{otherwise.} \end{cases}$$

The spatial derivative of the function can be expressed as

$$\Psi^{p,q}(r,\theta) = \sum_{i=0}^{M-p} \sum_{j=0}^{N-q} \delta_{ij} \Psi_{ij}^{p,q} T_i(r) T_j(\theta), \tag{25}$$

where p and q are orders of derivative with respect to r and θ, respectively.

The coefficients $\Psi_{ij}^{p,q}$ can be evaluated using the recurrence relation [25] as

$$\Psi_{(i-1)j}^{p,q} = \Psi_{(i+1)j}^{p,q} + 4i\Psi_{ij}^{(p-1),q}, \tag{26}$$

$$\Psi_{i(j-1)}^{p,q} = \Psi_{i(j+1)}^{p,q} + 4j\Psi_{ij}^{p,(q-1)}. \tag{27}$$

The displacement components u, v, w, ϕ_r and ϕ_θ are approximated as

$$(u, v, w, \phi_r, \phi_\theta) = \sum_{i=0}^{M} \sum_{j=0}^{N} \delta_{ij} (u, v, w, \phi_r, \phi_\theta)_{ij} T_i(r) T_j(\theta). \tag{28}$$

The algebraic eigenvalue problem is formulated by generating the appropriate equations from the equilibrium equations and the boundary conditions.

Clamped edge boundary condition on any edge is applied as

$$u = v = w = \phi_r = \phi_\theta = 0. \tag{29}$$

Simply supported radial edge implies

$$u = N_\theta = w = \phi_r = M_\theta = 0. \tag{30}$$

Free radial edge requires

$$N_{r\theta} = N_\theta = Q_\theta = M_{r\theta} = M_\theta = 0. \tag{31}$$

Simply supported circumferential edge is expressed as

$$N_r = v = w = M_r = \phi_\theta = 0. \tag{32}$$

Free circumferential edge gives

$$N_r = N_{r\theta} = Q_r = M_r = M_{r\theta} = 0. \tag{33}$$

For example, in case of the displacement component ϕ_r, which is always zero on the radial edges (for various combinations of boundary conditions in which all the edges are either simply supported or clamped), the corresponding algebraic equations introduced into the total set of equations are

$$\sum_{j=0}^{N} \delta_{ij}(-1)^j \phi_{r,ij} = 0 \quad \text{where } 0 \leqslant i \leqslant M \text{ for the edge } \theta = 0, \tag{34}$$

$$\sum_{j=0}^{N} \delta_{ij} \phi_{r,ij} = 0 \quad \text{where } 0 \leqslant i \leqslant M \text{ for the edge } \theta = 1. \tag{35}$$

Each of these two boundary conditions thus contributes total $(M+1)$ equations.

On the other hand, on the circumferential edges, $\phi_r = 0$ for the clamped edge and $M_r = 0$ for the simply supported edge.

In Chebyshev technique more algebraic equations are generated than the unknowns. Nath and Kumar [26] employed the linear regression analysis with least square error norms and obtained the unique solution. In the present paper the redundancy has been overcome by using linear interpolation for expressing the various combinations of boundary conditions.

For the clamped edge at $r = 0$, the displacement component ϕ_r along this edge is visualized as

$$\sum_{j=0}^{N} T_j(\theta) \sum_{i=0}^{M} \delta_{ij}(-1)^i \phi_{r,ij} = (B_2 \Phi_{r,00} + B_1 \Phi_{r,01}) T_N(\theta) + (A_2 \Phi_{r,00} + A_1 \Phi_{r,01}) T_{N-1}(\theta). \tag{36}$$

Here, $\Phi_{r,00}$ and $\Phi_{r,01}$ are ϕ_r values at the corner points $(r = 0, \theta = 0)$ and $(r = 0, \theta = 1)$, and have already been specified as zero by virtue of Eqs. (34), (35) above. A_1, A_2, B_1 and B_2 are the interpolation constants.

Then, in order to avoid redundancy in application of boundary condition and hence, in order to have non-singular system of equations, the corresponding boundary condition is applied as follows:

$$\sum_{i=0}^{M} \delta_{ij}(-1)^i \phi_{r,ij} = 0 \quad \text{where } 0 \leqslant j \leqslant N-2 \text{ for the edge } r = 0. \tag{37}$$

The above set of $(N-1)$ equations thus simply skips the equations corresponding to coefficients of the two highest order terms in Eq. (36).

Application of simply supported boundary condition on a circumferential edge is similarly accomplished in view of the fact that $\phi_\theta = 0$ on these edges.

Thus, application of boundary conditions (various combinations of fixed and simply supported, on all the four edges) corresponding to ϕ_r introduces total $2(M+N)$ equations.

The force boundary condition $M_{r\theta} = 0$ on the free radial edges of, for example, an SFSF plate is expressed as

$$rM_{r\theta} = \sum_{i=0}^{M} \sum_{j=0}^{N} \delta_{ij}(rM_{r\theta})_{ij} T_i(r) T_j(\theta) = 0. \tag{38}$$

$(rM_{r\theta})_{ij}$ are the coefficients in the expansion of $rM_{r\theta}$ obtained in terms of coefficients $(u, v, w, \phi_r \text{ and } \phi_\theta)_{ij}$.

Thus, exact application of $rM_{r\theta} = 0$ edge condition on the edge $\theta = 0$ would give $(M+1)$ equations as follows:

$$\sum_{j=0}^{N} \delta_{ij}(-1)^j (rM_{r\theta})_{ij} = 0 \quad \text{where } 0 \leqslant i \leqslant M \text{ for the edge } \theta = 0. \tag{39}$$

Here, the two equations in Eq. (39) for $(i = M, (M-1))$ are ignored in view of the minimax properties of Chebyshev polynomials [25]. Similarly, the two other boundary conditions $M_r = 0$ and $Q_r = 0$ are applied, each contributing $2(M-1)$ equations.

Each of the five equations (Appendix A) will give the following $(M+1)(N+1)$ equations:

$$F_{A,i'j'}(u, v, w, \phi_r, \phi_\theta)_{ij} - \eta F_{B,i'j'}(u, v, w, \phi_r, \phi_\theta)_{ij} = 0 \quad \text{where } 0 \leqslant i' \leqslant M \text{ and } 0 \leqslant j' \leqslant N. \tag{40}$$

Here, η is the eigenvalue so that $12\eta = N_{\text{cr}}$ for the buckling problem ω^2 for vibration problem $F_{A,i'j'}$ and $F_{B,i'j'}$ are linear functions of coefficients $(u, v, w, \phi_r \text{ and } \phi_\theta)_{ij}$.

The coefficients of $T_{i'}(r)$ for $i' > M$ caused by multiplication with various powers of r are ignored. To accommodate the $2(M+N)$ boundary condition equations for each degree of freedom, the coefficients of two highest order Chebyshev polynomials in both r and θ dimensions are ignored. The theoretical basis for ignoring these equations again is the minimax property of Chebyshev polynomials [25]. This produces exactly $(M+1)(N+1)$ equations for each of these three degrees of freedom as follows:

$$\underbrace{[(M-2)+1][(N-2)+1]}_{\substack{\text{From equilibrium} \\ \text{equations}}} + \underbrace{2(M+N)}_{\substack{\text{From boundary} \\ \text{conditions}}} = (M+1)(N+1). \tag{41}$$

Each of these five sets of $(M+1)(N+1)$ equations in terms of the $5(M+1)(N+1)$ number of coefficients and the eigenvalue are arranged as

$$[A'_{\text{dof}}]\{X\} - \eta[B'_{\text{dof}}]\{X\} = 0, \quad \text{dof} = u, v, w, \phi_r, \phi_\theta. \tag{42}$$

Here

$$\{X\} = \begin{Bmatrix} U \\ V \\ W \\ \Phi_r \\ \Phi_\theta \end{Bmatrix}. \tag{43}$$

$[A'_{\text{dof}}]$ and $[B'_{\text{dof}}]$ are rectangular matrices of dimension $(M+1)(N+1)$ by $5(M+1)(N+1)$. The boundary conditions are included by having the lowest $2(M+N)$ rows of $[B'_{\text{dof}}]$ as null vectors and those of $[A'_{\text{dof}}]$ obtained from left-hand sides of the boundary condition equations corresponding to a particular degree of freedom.

Assembling the five matrix equations in Eq. (42), we get the following eigenvalue problem:

$$[A]^{-1}[B]\{X\} = \left(\frac{1}{\eta}\right)\{X\}. \tag{44}$$

$[A]$ and $[B]$ are square matrices given by

$$[A] = \begin{bmatrix} A'_u \\ A'_v \\ A'_w \\ A'_{\phi_r} \\ A'_{\phi_\theta} \end{bmatrix}, \quad [B] = \begin{bmatrix} B'_u \\ B'_v \\ B'_w \\ B'_{\phi_r} \\ B'_{\phi_\theta} \end{bmatrix}. \tag{45}$$

The eigenvalue problem in Eq. (44) is solved using the procedure given by Wilkinson and Reinsch [27].

4. Results and discussions

The present study gives the solutions of the eigenvalue problems resulting from buckling and free vibration analyses of moderately thick sector plates. The effects of orthotropy, number of layers, annularity (α), thickness ratio (λ), sector angle (Θ), rotary inertia and boundary conditions are studied. The boundary conditions considered here are various combinations of simply supported (S), clamped (C) and free (F) edges. The combinations are named in the order shown in Fig. 1.

Every layer is assumed to be cylindrically orthotropic. Thus with fibers oriented in radial direction, the required material properties are as follows unless otherwise specified.

$$E_r/E_\theta = 40, \quad v_{r\theta} = 0.25, \quad G_{r\theta} = G_{zr} = 0.6E_\theta, \quad G_{\theta z} = 0.5E_\theta.$$

The rotation of material axes by 90° about the global z-axis gives the material constants for a layer having fibers oriented in circumferential direction. Thus a 10 layer ($n = 10$) cross-ply antisymmetric plate denoted in rectangular domain by $(0°/90°)_5$ is represented here by $(R/C)_5$ with 'R' representing radial and 'C' representing circumferential orientation of the fiber.

Results for convergence with respect to the degrees M and N of the series expansions are given in Tables 1–3 in terms of vibration frequencies and critical buckling loads. Two more terms were used in r dimension than the θ dimension since the equilibrium equations were multiplied with r^2 in order to remove r from the denominators of various terms.

Tables 1 and 2 give the convergence of the lowest six frequencies for the vibration problem. Convergence to five significant digits is achieved in the fundamental frequency with ($M = 11, N = 9$) in Table 1 for SSSS plate and to four significant digits with ($M = 15, N = 13$) in Table 2 for FSFS plate.

Table 3 gives the convergence of lowest critical buckling load for the boundary conditions SSSS. Convergence to five significant digits is achieved with ($M = 17, N = 15$).

In the present study total 168 terms ($M = 13, N = 11$) gave satisfactory results for all the combinations of various parameters, and these are thus were taken as default values of M and N.

Table 1
Convergence results for non-dimensional natural frequencies $\{\omega(1-\alpha)^2 r_o^2\sqrt{\rho/E_\theta h^2}\}$ of SSSS plate with respect to number of terms in series expansion, $n = 10$, $(R/C)_5$, $\alpha = 0.1$, $\lambda = 0.2$, $\Theta = \pi/3$

$(M+1)$ $(N+1)$	Mode sequence number 1	2	3	4	5	6
80	12.36402	15.25554	17.95893	22.99828	25.19537	25.66754
120	12.36342	15.25569	17.90371	22.99736	25.19380	25.66610
168	12.36324	15.25601	17.87369	22.99695	25.19393	25.66603
224	12.36320	15.25606	17.84693	22.99687	25.19384	25.66606
288	12.36319	15.25599	17.82245	22.99685	25.19367	25.66609
324	12.36319	15.25589	17.80329	22.99685	25.19351	25.66610

Table 2
Convergence results for non-dimensional natural frequencies $\{\omega(1-\alpha)^2 r_o^2\sqrt{\rho/E_\theta h^2}\}$ of FSFS plate with respect to number of terms in series expansion, $n = 10$, $(R/C)_5$, $\alpha = 0.1$, $\lambda = 0.2$, $\Theta = \pi/3$

$(M+1)$ $(N+1)$	Mode sequence number 1	2	3	4	5	6
80	6.760439	15.77599	16.06734	17.17089	18.44083	26.06850
120	6.765566	15.91724	15.93375	17.16948	18.44208	26.60386
168	6.767594	15.87374	15.97915	17.17088	18.44259	26.38041
224	6.768336	16.00553	16.37862	17.17111	18.44282	26.17791
288	6.768574	16.01627	16.10377	17.16955	18.44291	26.11672
324	6.768627	16.02031	16.30925	17.17047	18.44294	26.40108

Table 3
Convergence results for non-dimensional critical buckling load $\{N_{cr}(1-\alpha)^2 r_o^2/E_\theta h^3\}$ of SSSS plate with respect to number of terms in series expansion, $n = 10$, $(R/C)_5$, $\mu = 0.1$, $\lambda = 0.2$, $\Theta = \pi/3$

$(M+1)$ $(N+1)$	$\{N_{cr}(1-\alpha)^2 r_o^2/E_\theta h^3\}$
80	7.117115
120	7.117316
168	7.117687
224	7.117802
288	7.117783
324	7.117724

The developed methodology is further validated in Table 4 by comparing the results of almost square sector plates with results for free vibration of laminated square plates given in Table 3 of the paper by Khdeir [1] using first-order shear deformation theory. The results shown are for edge conditions SSSS, SSCS, FSFS and FSSS. The two sets agree well within the range of 1%.

Table 5 gives the comparison of critical buckling loads of SSSS and SSCS isotropic sector plates with those given by Wang and Xiang [21]. Again, the two sets agree within the range of 1%.

Table 6 gives the comparison of non-dimensional natural frequencies of SSSS and CFCF isotropic sector plates with those given by Ramakrishnan and Kunukkasseril [8] using exact analysis for thin plates, and, by Xiang et al. using Rayleigh–Ritz method (RRM) [14] and Liew and Liu [23] using differential quadrature method (DQM) for the Mindlin plates. To facilitate this comparison, in-plane inertia was ignored. There is good agreement among these results.

Table 4
Comparison of results for natural frequencies of almost square sector plates, $\alpha = 0.99$, $\lambda = 0.001$, $\Theta = 0.01$, $n = 2$, $(R/C), E_r/E_\theta = 1$, $v_{r\theta} = 0.3$, $G_{r\theta} = G_{\theta z} = G_{zr} = 0.5/(1+v_{r\theta})$

n	E_r/E_θ	$\{\omega_1(1-\alpha)^2 r_o^2\sqrt{\rho/E_\theta h^2}\}$			
		SSSS	SSCS	FSFS	FSSS
$2, (R/C)$	2 Present	6.614	7.761	3.339	4.043
	Ref. [1]	6.82	7.739	3.309	4.035
	10	7.803	9.375	4.627	5.119
		7.766	9.352	4.584	5.104
	20	8.852	10.683	5.563	5.958
		8.810	10.658	5.511	5.940
	30	9.741	11.742	6.309	6.645
		9.695	11.713	6.251	6.624
	40	10.522	12.640	6.944	7.237
		10.473	12.610	6.881	7.215
$10, (R/C)_5$	2	6.781	7.975	3.482	4.164
		6.749	7.953	3.451	4.156
	10	9.945	11.941	6.366	6.711
		9.899	11.911	6.307	6.691
	20	12.523	14.795	8.438	8.660
		12.466	14.758	8.365	8.634
	30	14.391	16.696	9.882	10.043
		14.327	16.652	9.800	10.014
	40	15.847	18.094	10.987	11.110
		15.779	18.044	10.900	11.079

Table 5
Comparison of results for critical buckling loads of isotropic sector plates, $\alpha = 0$, $\lambda = 0.2$, $n = 2$, (R/C), $E_r/E_\theta = 1$, $v_{r\theta} = 0.3$, $G_{r\theta} = G_{\theta z} = G_{zr} = 0.5/(1+v_{r\theta})$

Θ	$\{12(1-\upsilon^2)N_{cr}(1-\alpha)^2 r_o^2/E_\theta h^3\}$			
	SSSS		SSCS	
	Present	Ref. [21]	Present	Ref. [21]
$\pi/6$	46.05	45.98	49.72	49.24
$\pi/3$	27.06	26.90	33.70	33.88
$\pi/2$	19.34	19.14	27.08	27.14

Table 6

Comparison of results for non-dimensional natural frequencies $\{\omega r_o^2\sqrt{12(1-\nu^2)\rho/E_\theta h^2}\}$ of isotropic sector plates, $n=2$, (R/C), $E_r/E_\theta=1$, $\nu_{r\theta}=0.3$, $G_{r\theta}=G_{\theta z}=G_{zr}=0.5/(1+\nu_{r\theta})$, in-plane inertia is ignored

Θ	α	λ	Boundary condition	Method	Mode sequence number				
					1	2	3	4	5
$\pi/4$	0.5	0.005	SSSS	Chebyshev	68.358	150.877	189.433	278.057	283.224
	0.5	0.005	SSSS	DQM [23]	68.357	150.880	189.430	278.030	283.220
	0.5	Thin plate	SSSS	Exact [8]	68.380	150.960	189.610	278.460	283.590
$\pi/3$	0.5	0.1	SSSS	Chebyshev	51.026	88.606	138.868	140.979	171.622
	0.5	0.1	SSSS	DQM [23]	50.982	88.486	138.600	140.700	171.230
	0.5	0.1	SSSS	RRM [14]	51.025	88.530	138.940	140.890	171.630
	0.5	0.2	SSSS	Chebyshev	42.090	67.442	97.690	99.080	116.302
	0.5	0.2	SSSS	DQM [23]	41.995	67.237	97.320	98.704	115.820
	0.5	0.2	SSSS	RRM [14]	42.066	67.394	97.698	99.040	116.290
$\pi/4$	0.4	0.005	CFCF	Chebyshev	61.001	74.894	131.285	167.355	189.626
	0.4	0.005	CFCF	DQM [23]	61.112	75.075	132.570	169.190	190.410
	0.4	Thin plate	CFCF	Exact [8]	61.160	75.150	132.860	169.320	190.260
$\pi/2$	0.4	0.1	CFCF	Chebyshev	51.396	53.752	63.775	84.498	114.824
	0.4	0.1	CFCF	DQM [23]	51.304	53.649	63.660	84.339	114.590
	0.4	0.1	CFCF	RRM [14]	51.406	53.760	63.785	84.497	114.820
	0.4	0.2	CFCF	Chebyshev	37.595	39.126	46.205	60.754	77.898
	0.4	0.2	CFCF	DQM [23]	37.446	38.975	46.046	60.568	77.541
	0.4	0.2	CFCF	RRM [14]	37.597	39.126	46.202	60.761	77.895

Table 7 gives the effect of number of layers n and moduli ratio E_r/E_θ on lowest six non-dimensional natural frequencies of sector plates with four combinations of clamped and simply supported edges. Table 8 gives the same effect with one or two edges being free. There is increase in frequencies with the increase in non-homogeneity in terms of n as well as E_r/E_θ. It can be seen there is repetition of certain frequencies over variation of the three parameters—boundary conditions, number of layers n and moduli ratio E_r/E_θ. These repeating modes were found to be those of uncoupled in-plane vibrations. To verify this, all inertia elements in the equations of motion other than I_0 in the equations of motion in r and θ directions were made zero for SSSS plate in Table 7 and for CSFS plates in Table 8 for the results corresponding to $n=10$ and $E_r/E_\theta=40$. The non-dimensional frequencies of the resulting two-dimensional problem are thus shown in bold along with those of the actual five-dimensional problem for comparison. The frequencies of uncoupled in-plane vibrations have been shown italicized. These frequencies remain unchanged since the extensional stiffnesses A_{ij} remain unaffected by the variations in n and E_r/E_θ.

The effects of thickness ratio λ and annularity α on the frequencies are shown in Table 9. Here, when α is increased from 0.1 to 0.5, the frequencies increase in the absolute sense after removal of the factor $(1-\alpha)^2$ from the non-dimensionalization factor $\{(1-\alpha)^2 r_o^2\sqrt{\rho/E_\theta h^2}\}$. These results also show that the non-dimensional frequencies decrease with increasing thickness parameter λ, but the absolute frequencies actually increase with increasing thickness.

Fig. 2 shows the effect of sector angle Θ on the fundamental natural frequency. It can be seen that the frequency decreases with increasing span in terms of Θ in all plates except for the one having radial edges free—SFSF.

Table 7

Effect of n and E_r/E_θ on non-dimensional natural frequencies $\{\omega(1-\alpha)^2 r_o^2 \sqrt{\rho/E_\theta h^2}\}$ of sector plates with combinations of clamped and simply supported edges, $\alpha = 0.1$, $\lambda = 0.2$, $\Theta = \pi/3$

Boundary condition	n, fiber orientation	E_r/E_θ	Mode sequence number					
			1	2	3	4	5	6
CCCC	2, (R/C)	2	13.095	20.964	21.354	25.652	25.789	29.408
		10	14.629	22.846	22.952	31.576	31.684	32.197
		20	15.395	23.819	23.868	32.859	32.896	33.275
		30	15.829	24.350	24.457	33.504	33.679	33.909
		40	16.113	24.720	24.846	33.913	34.216	34.331
	10, $(R/C)_5$	2	13.305	21.274	21.676	25.679	25.990	29.824
		10	15.717	24.181	24.480	33.374	34.068	34.163
		20	16.445	25.147	25.411	34.375	35.065	35.308
		30	16.747	25.579	25.844	34.778	35.485	35.826
		40	16.912	25.827	26.103	34.996	35.719	36.125
CSCS	2, (R/C)	2	10.483	12.363	18.795	19.461	21.014	22.997
		10	11.999	12.363	20.658	21.606	22.997	29.957
		20	12.363	12.920	21.829	22.839	22.997	31.355
		30	12.363	13.533	22.592	22.997	23.603	32.171
		40	12.363	13.983	22.997	23.141	24.130	32.722
	10, $(R/C)_5$	2	10.706	12.363	19.425	19.843	20.796	22.997
		10	12.363	13.613	22.927	22.997	23.751	32.489
		20	12.363	14.844	22.997	24.274	25.068	33.719
		30	12.363	15.454	22.997	24.907	25.643	33.719
		40	12.363	15.828	22.997	25.280	25.971	33.719
SSSS	2,(R/C)	2	8.973	11.506	12.363	17.927	18.222	22.286
		10	10.348	12.363	14.691	20.175	20.177	22.997
		20	11.381	12.363	15.876	21.546	21.585	22.997
		30	12.123	12.363	16.519	22.413	22.513	22.997
		40	12.363	12.695	16.962	22.997	23.021	23.176
	2, $(R/C)_5$	2	9.258	11.457	12.363	18.334	18.792	22.149
		10	12.336	12.363	14.983	22.524	22.937	22.997
		20	12.363	13.914	16.487	22.997	24.082	24.529
		30	12.363	14.742	17.314	22.997	24.787	25.251
[a]		**40**	**12.363**	**15.256**	**17.874**	**22.997**	**25.194**	**25.666**
		40	**12.363**	**17.868**	**22.997**	**32.605**	**33.719**	**44.512**
CSSS	2, (R/C)	2	8.978	11.514	12.363	17.943	18.222	22.286
		10	10.410	12.363	15.532	20.179	20.211	22.997
		20	11.481	12.363	18.290	21.598	21.624	22.997
		30	12.239	12.363	20.399	22.535	22.569	22.997
		40	12.363	12.817	22.042	22.997	23.203	23.359
	10, $(R/C)_5$	2	9.263	11.465	12.363	18.354	18.792	22.149
		10	12.363	12.381	15.731	22.603	22.938	22.997
		20	12.363	13.980	18.789	22.997	24.164	24.532
		30	12.363	14.811	21.255	22.997	24.861	25.256
		40	12.363	15.321	22.997	23.409	25.260	25.672

[a]Only in-plane inertia acting.

Table 8

Effect of n and E_r/E_θ on non-dimensional natural frequencies $\{\omega(1-\alpha)^2 r_o^2\sqrt{\rho/E_\theta h^2}\}$ of sector plates with combinations of clamped, simply supported and free edges, $\alpha = 0.1$, $\lambda = 0.2$, $\Theta = \pi/3$

Boundary condition	n, fiber orientation	E_r/E_θ	Mode sequence number					
			1	2	3	4	5	6
FCFC	2, (R/C)	2	6.289	13.224	13.336	15.174	17.455	21.238
		10	7.537	14.784	15.291	20.457	22.538	23.350
		20	8.214	15.539	16.579	23.759	24.030	24.607
		30	8.629	15.999	17.369	24.869	24.980	26.604
		40	8.918	16.323	17.906	25.352	25.688	27.767
	10, $(R/C)_5$	2	6.416	13.447	13.622	15.101	17.651	21.578
		10	8.371	15.994	16.578	21.082	25.010	25.771
		20	9.154	16.879	17.879	25.862	26.143	27.942
		30	9.555	17.322	18.569	26.649	28.812	29.103
		40	9.814	17.604	19.022	26.967	29.328	29.493
SFSF	2, (R/C)	2	3.013	3.158	4.983	6.001	10.589	13.950
		10	4.154	4.487	6.318	7.361	12.902	15.900
		20	4.728	5.316	6.699	8.130	14.117	16.666
		30	5.103	5.857	6.960	8.509	14.875	17.158
		40	5.382	6.265	7.166	8.745	15.409	17.508
	10, $(R/C)_5$	2	3.058	3.239	5.021	6.052	10.900	14.502
		10	4.948	5.009	7.140	7.450	14.949	17.314
		20	5.838	6.139	7.734	8.379	16.384	18.078
		30	6.270	6.906	8.060	8.821	17.026	18.479
		40	6.537	7.475	8.273	9.086	17.392	18.741
CSFS	2, (R/C)	2	0.899	3.369	10.708	10.731	12.056	16.725
		10	0.899	4.215	12.183	12.743	16.238	16.725
		20	0.899	4.754	13.411	13.877	16.725	18.854
		30	0.899	5.147	14.317	14.615	16.725	20.804
		40	0.899	5.460	15.022	15.151	16.725	22.375
	10, $(R/C)_5$	2	0.899	3.455	10.977	11.371	11.672	16.725
		10	0.899	5.086	14.632	14.992	15.766	16.725
		20	0.899	6.038	16.071	16.725	16.864	18.620
		30	0.899	6.627	16.725	16.773	17.831	20.958
		40	**0.899**	**7.050**	**16.725**	**17.208**	**18.432**	**23.018**
[a]		**40**	**0.899**	**16.725**	**22.960**	**27.770**	**33.875**	**38.691**
SSFS	2, (R/C)	2	0.899	3.368	10.708	10.722	12.051	16.725
		10	0.899	4.196	12.002	12.743	15.508	16.725
		20	0.899	4.711	13.039	13.873	16.686	16.725
		30	0.899	5.083	13.798	14.605	16.725	17.286
		40	0.899	5.377	14.388	15.133	16.725	17.681
	10, $(R/C)_5$	2	0.899	3.454	10.977	11.363	11.665	16.725
		10	0.899	5.068	14.632	14.704	15.264	16.725
		20	0.899	5.967	16.067	16.192	16.725	16.960
		30	0.899	6.482	16.725	16.758	16.945	17.887
		40	0.899	6.822	16.725	17.176	17.425	18.477

[a]Only in-plane inertia acting.

Table 9

Effect of λ and α on non-dimensional natural frequencies $\{\omega(1-\alpha)^2 r_o^2\sqrt{\rho/E_\theta h^2}\}$ of sector plates with combinations of clamped, simply supported and free edges, $\Theta=\pi/3$, $n=10$, $(R/C)_5$, $E_r/E_\theta=40$

Boundary condition	λ	α	Mode sequence number					
			1	2	3	4	5	6
CCCC	0.2	0.1	16.912	25.827	26.103	34.996	35.719	36.125
		0.5	6.169	8.335	10.952	11.103	12.469	13.916
	0.1	0.1	31.063	48.085	48.540	66.492	67.688	67.778
		0.5	11.611	15.621	20.843	21.098	23.686	26.774
	0.01	0.1	70.061	131.034	132.817	209.955	213.194	213.588
		0.5	30.413	42.818	66.041	75.149	82.599	95.771
SSSS	0.2	0.1	12.363	15.256	17.874	22.997	25.194	25.666
		0.5	4.526	5.690	6.190	7.847	8.122	10.841
	0.1	0.1	23.542	24.726	35.753	43.973	44.807	45.994
		0.5	9.048	9.240	12.380	14.084	15.702	20.102
	0.01	0.1	32.950	80.714	81.747	146.511	149.461	152.020
		0.5	14.103	24.953	45.986	50.504	56.406	72.422
CSSS	0.2	0.1	12.363	15.321	22.997	23.409	25.260	25.672
		0.5	5.733	6.190	8.156	10.851	10.974	12.224
	0.1	0.1	23.713	24.726	44.256	44.819	45.994	46.880
		0.5	9.734	12.380	14.413	20.275	20.332	23.322
	0.01	0.1	32.957	81.140	81.747	149.005	149.461	152.020
		0.5	19.447	27.324	46.263	61.078	65.585	72.422
CSCS	0.2	0.1	12.363	15.828	22.997	25.280	25.971	33.719
		0.5	5.958	6.190	8.297	10.852	11.076	12.224
	0.1	0.1	24.726	26.539	45.052	45.994	46.795	64.262
		0.5	10.825	12.380	15.126	20.474	20.804	23.396
	0.01	0.1	44.831	96.755	99.805	169.006	171.967	173.459
		0.5	28.395	35.281	54.054	74.190	78.093	81.545
FCFC	0.2	0.1	9.814	17.604	19.022	26.967	29.328	29.493
		0.5	2.874	4.675	5.209	7.103	7.515	8.178
	0.1	0.1	16.125	31.069	32.864	48.827	52.400	53.757
		0.5	4.705	8.237	9.361	14.204	14.913	15.066
	0.01	0.1	26.729	65.853	67.176	126.189	127.602	130.000
		0.5	7.917	18.436	20.707	35.562	38.880	38.945
SFSF	0.2	0.1	6.537	7.475	8.273	9.086	17.392	18.741
		0.5	2.681	4.177	4.784	4.877	6.003	7.566
	0.1	0.1	9.784	11.644	14.964	18.173	29.713	31.911
		0.5	5.367	8.037	8.198	8.362	12.007	12.229
	0.01	0.1	13.016	14.982	50.913	54.671	57.529	112.487
		0.5	12.818	13.076	18.495	35.188	49.728	50.084
CFCF	0.2	0.1	8.496	9.512	11.746	17.521	19.131	21.665
		0.5	5.197	5.274	6.090	7.853	10.313	10.392
	0.1	0.1	14.765	15.855	23.496	31.538	33.618	40.584
		0.5	9.914	9.988	12.181	13.442	19.074	19.856
	0.01	0.1	26.826	28.053	70.637	76.370	76.997	134.590
		0.5	27.750	27.891	30.897	43.807	67.655	73.720

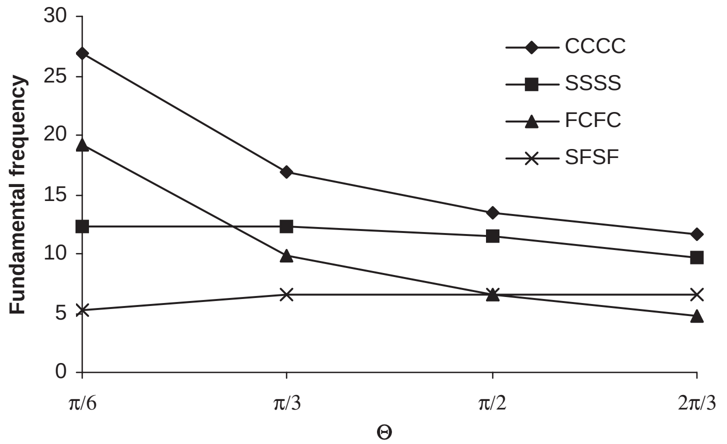

Fig. 2. Effect of Θ on non-dimensional fundamental natural frequencies $\{\omega_1(1-\alpha)^2 r_o^2\sqrt{\rho/E_\theta h^2}\}$ of sector plates with combinations of clamped, simply supported and free edges, $\lambda = 0.2$, $\alpha = 0.1$, $n = 10$, $(R/C)_5$, $E_r/E_\theta = 40$.

Table 10
Effect of n and E_r/E_θ on non-dimensional isotropic critical buckling load $\{N_{cr}(1-\alpha)^2 r_o^2/E_\theta h^3\}$ of sector plates with combinations of clamped and simply supported edges, $\alpha = 0.1$, $\lambda = 0.2$, $\Theta = \pi/3$

n, fiber orientation	E_r/E_θ	Boundary conditions			
		CCCC	CSCS	SSSS	CSSS
2, (R/C)	2	4.8394	3.8754	2.6736	3.2610
	10	6.1226	5.0890	3.6089	4.2955
	20	6.8289	5.8595	4.2939	4.9863
	30	7.2609	6.3677	4.8057	5.4689
	40	7.5578	6.7347	5.2128	5.8354
10, $(R/C)_5$	2	4.9862	3.9990	2.7787	3.3759
	10	7.1371	6.0918	4.7754	5.3765
	20	7.9346	7.0461	5.9831	6.4221
	30	8.2908	7.5238	6.6706	6.9926
	40	8.4942	7.8166	7.1177	7.3620

The effects of number of layers n and moduli ratio E_r/E_θ on non-dimensional isotropic critical buckling loads of sector plates with four combinations of clamped and simply supported edges are shown in Table 10. There is increase in critical buckling load with the increase in non-homogeneity in terms of n as well as E_r/E_θ.

Table 11 shows the effects of λ, α and Θ on the critical buckling loads. It can be noticed that the variation of non-dimensional critical buckling load is similar to the variation of non-dimensional frequencies. The non-dimensional critical buckling loads decrease with increasing λ and α but the absolute critical buckling loads increase with increasing λ and α. The critical buckling load always decreases with increasing span in terms of Θ. This decrease is largest for the most flexible edge condition SSSS and smallest for the stiffest edge condition CCCC.

Table 12 gives the effect of ignoring in-plane and rotary inertias on the lowest 10 natural frequencies of SSSS and CCCC plates. In CCCC plates the ignoring of in-plane inertia and then

Table 11
Effect of λ, α and Θ on non-dimensional isotropic critical buckling load $\{N_{cr}(1-\alpha)^2 r_o^2/E_\theta h^3\}$ of sector plates with combinations of clamped and simply supported edges, $n = 10$, $(R/C)_5$, $E_r/E_\theta = 40$

λ	α	Θ	Boundary conditions			
			CCCC	CSCS	SSSS	CSSS
0.2	0.1	$\pi/6$	8.905	8.623	8.275	8.348
	0.1	$\pi/3$	8.494	7.817	7.118	7.362
	0.1	$\pi/2$	8.145	7.163	6.298	6.648
	0.1	$2\pi/3$	7.861	6.655	5.785	6.155
	0.5	$\pi/6$	2.768	2.661	2.552	2.577
	0.5	$\pi/3$	2.681	2.481	2.345	2.397
	0.5	$\pi/2$	2.631	2.404	2.312	2.347
	0.5	$2\pi/3$	2.606	2.371	2.321	2.331
0.1	0.1	$\pi/6$	32.209	29.719	25.258	26.974
	0.1	$\pi/3$	27.481	23.027	16.975	19.716
	0.1	$\pi/2$	24.312	18.708	12.915	15.542
	0.1	$2\pi/3$	22.309	15.894	10.984	13.132
	0.5	$\pi/6$	10.103	9.105	7.721	8.245
	0.5	$\pi/3$	9.185	7.283	6.198	6.648
	0.5	$\pi/2$	8.816	6.664	6.179	6.316
	0.5	$2\pi/3$	8.610	6.481	6.198	6.261
0.01	0.1	$\pi/6$	274.543	202.934	82.371	132.804
	0.1	$\pi/3$	114.687	73.694	32.817	50.205
	0.1	$\pi/2$	75.696	44.035	20.281	30.982
	0.1	$2\pi/3$	60.185	32.014	16.052	22.623
	0.5	$\pi/6$	85.671	63.041	24.010	40.371
	0.5	$\pi/3$	51.997	21.095	14.305	16.398
	0.5	$\pi/2$	47.234	16.275	14.930	15.064
	0.5	$2\pi/3$	45.521	15.800	14.305	14.825

Table 12
Effect of ignoring in-plane and rotary inertia on non-dimensional natural frequencies $\{\omega(1-\alpha)^2 r_o^2 \sqrt{\rho/E_\theta h^2}\}$ of CCCC and SSSS sector plates, $\lambda = 0.2$, $\alpha = 0.1$, $\Theta = \pi/3$, $n = 10$, $(R/C)_5$, $E_r/E_\theta = 40$

Mode sequence number	CCCC			SSSS		
	Inertias ignored			Inertias ignored		
	None	In-plane	In-plane and rotary	None	In-plane	In-plane and rotary
1	16.91189	16.91204	16.91984	12.36324	15.26163	15.3217
2	25.82682	25.82768	25.87536	15.25601	25.19409	25.25778
3	26.10315	26.10473	26.18344	17.87369	25.66957	25.73206
4	34.99554	34.9962	35.02473	22.99695	34.49812	34.5652
5	35.71939	35.72054	35.77799	25.19393	35.3995	35.46227
6	36.12453	36.12743	36.21512	25.66603	36.05172	36.10947
7	43.95235	43.95306	43.99203	32.60073	43.61446	43.67999
8	44.99661	44.99819	45.06566	33.71909	44.79206	44.85464
9	45.71234	45.71438	45.78685	34.49660	45.62321	45.68109
10	46.35041	46.35245	46.41758	35.39660	46.29722	46.35353

additionally rotary inertia raise all the frequencies by less than 0.5%. In SSSS plates ignoring of rotary inertia raises all the frequencies by less than 0.5%, but including in-plane inertia introduces certain frequencies of uncoupled in-plane vibrations which are absent from the other two cases considered here.

Thus, frequencies and buckling loads increase with increasing constraint in terms of boundary conditions and non-homogeneity of the material.

5. Conclusions

The eigenvalue problem of free vibration and buckling of moderately thick laminated sector plates has been solved using Chebyshev polynomials. The formulation facilitates for the first time a simple and efficient solution in the sense that an expansion of 14 terms in r dimensions and 12 terms in θ dimension was found satisfactory for most of the given results. The methodology developed proved to be robust and efficient for all combinations of clamped and simply supported edge conditions and also gives correct free vibration results for two opposite edges being free and the other two either simply supported or clamped. The results are found to be in good agreement with those given in the literature.

Appendix A

The displacement equations of equilibrium are given below.

- u direction

$$
\begin{aligned}
&A_{11}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 u}{\partial r^2}+2A_{16}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2 u}{\partial r\partial\theta}+A_{66}\frac{1}{\Theta^2}\frac{\partial^2 u}{\partial\theta^2}+A_{11}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial u}{\partial r}-A_{22}u\\
&+A_{16}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 v}{\partial r^2}+(A_{12}+A_{66})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2 v}{\partial r\partial\theta}+A_{26}\frac{1}{\Theta^2}\frac{\partial^2 v}{\partial\theta^2}\\
&-A_{26}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial v}{\partial r}-(A_{22}+A_{66})\frac{1}{\Theta}\frac{\partial v}{\partial\theta}-A_{26}v\\
&+B_{11}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2\phi_r}{\partial r^2}+2B_{16}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2\phi_r}{\partial r\partial\theta}+B_{66}\frac{1}{\Theta^2}\frac{\partial^2\phi_r}{\partial\theta^2}\\
&+B_{11}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial\phi_r}{\partial r}-B_{22}\phi_r\\
&+B_{16}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2\phi_\theta}{\partial r^2}+(B_{12}+B_{66})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2\phi_\theta}{\partial r\partial\theta}\\
&+B_{26}\frac{1}{\Theta^2}\frac{\partial^2\phi_\theta}{\partial\theta^2}-B_{26}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial\phi_\theta}{\partial r}-(B_{22}+B_{66})\frac{1}{\Theta}\frac{\partial\phi_\theta}{\partial\theta}-B_{26}\phi_\theta\\
&\quad=\{r(1-\alpha)+\alpha\}^2 r_o^2\left(I_0\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}+I_1\frac{\mathrm{d}^2\phi_r}{\mathrm{d}t^2}\right).
\end{aligned}
\tag{A.1}
$$

- v direction

$$
\begin{aligned}
&A_{16}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 u}{\partial r^2}+(A_{12}+A_{66})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2 u}{\partial r\partial\theta}+A_{26}\frac{1}{\Theta^2}\frac{\partial^2 u}{\partial\theta^2}\\
&+(2A_{16}+A_{26})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial u}{\partial r}+(A_{22}+A_{66})\frac{1}{\Theta}\frac{\partial u}{\partial\theta}\\
&-A_{26}u+A_{66}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 v}{\partial r^2}+2A_{26}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2 v}{\partial r\partial\theta}+A_{22}\frac{1}{\Theta^2}\frac{\partial^2 v}{\partial\theta^2}\\
&-A_{66}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial v}{\partial r}-A_{66}v\\
&+B_{16}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2\phi_r}{\partial r^2}+(B_{12}+B_{66})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2\phi_r}{\partial r\partial\theta}+B_{26}\frac{1}{\Theta^2}\frac{\partial^2\phi_r}{\partial\theta^2}\\
&+(2B_{16}+B_{26})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial\phi_r}{\partial r}\\
&+(B_{22}+B_{66})\frac{1}{\Theta}\frac{\partial\phi_r}{\partial\theta}-B_{26}\phi_r+B_{66}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2\phi_\theta}{\partial r^2}\\
&+2B_{26}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2\phi_\theta}{\partial r\partial\theta}+B_{22}\frac{1}{\Theta^2}\frac{\partial^2\phi_\theta}{\partial\theta^2}-B_{66}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial\phi_\theta}{\partial r}-B_{66}\phi_\theta\\
&=\{r(1-\alpha)+\alpha\}^2 r_o^2\left(I_0\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}+I_1\frac{\mathrm{d}^2\phi_r}{\mathrm{d}t^2}\right).
\end{aligned}
\tag{A.2}
$$

- w direction

$$
\begin{aligned}
&k^2A_{55}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 w}{\partial r^2}+2k^2A_{45}(1-\alpha)\frac{\partial^2 w}{\partial r\partial\theta}+k^2A_{44}\frac{1}{\Theta^2}\frac{\partial^2 w}{\partial\theta^2}+k^2A_{55}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial w}{\partial r}\\
&+k^2A_{55}\frac{\{r(1-\alpha)+\alpha\}^2 r_o}{(1-\alpha)}\frac{\partial\phi_r}{\partial r}+k^2A_{45}\frac{\{r(1-\alpha)+\alpha\}r_o}{\Theta}\frac{\partial\phi_r}{\partial\theta}+k^2A_{55}\{r(1-\alpha)+\alpha\}r_o\phi_r\\
&+k^2A_{45}\frac{\{r(1-\alpha)+\alpha\}^2 r_o}{(1-\alpha)}\frac{\partial\phi_\theta}{\partial r}+k^2A_{45}\frac{\{r(1-\alpha)+\alpha\}r_o}{\Theta}\frac{\partial\phi_\theta}{\partial\theta}+k^2A_{44}\{r(1-\alpha)+\alpha\}r_o\phi_\theta\\
&+N_{\mathrm{ip}}\left[\left(\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 w}{\partial r^2}\right)+\left(\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial w}{\partial r}\right)+\frac{1}{\Theta^2}\frac{\partial^2 w}{\partial\theta^2}\right]\\
&=\{r(1-\alpha)+\alpha\}^2 r_o^2 I_0\frac{\mathrm{d}^2 w}{\mathrm{d}t^2}.
\end{aligned}
\tag{A.3}
$$

- ϕ_r direction

$$
B_{11}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 u}{\partial r^2}+2B_{16}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2 u}{\partial r\partial\theta}+B_{66}\frac{1}{\Theta^2}\frac{\partial^2 u}{\partial\theta^2}+B_{11}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial u}{\partial r}-B_{22}u
$$

$$+ B_{16}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 v}{\partial r^2} + (B_{12}+B_{66})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2 v}{\partial r\partial\theta} + B_{26}\frac{1}{\Theta^2}\frac{\partial^2 v}{\partial\theta^2} - B_{26}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial v}{\partial r}$$
$$- (B_{22}+B_{66})\frac{1}{\Theta}\frac{\partial v}{\partial\theta} - B_{26}v - k^2A_{55}\frac{\{r(1-\alpha)+\alpha\}^2 r_o}{(1-\alpha)}\frac{\partial w}{\partial r} - k^2A_{45}\frac{\{r(1-\alpha)+\alpha\}r_o}{\Theta}\frac{\partial w}{\partial\theta}$$
$$+ D_{11}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2\phi_r}{\partial r^2} + 2D_{16}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2\phi_r}{\partial r\partial\theta} + D_{66}\frac{1}{\Theta^2}\frac{\partial^2\phi_r}{\partial\theta^2} + D_{11}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial\phi_r}{\partial r}$$
$$- D_{22}\phi_r - k^2A_{55}\{r(1-\alpha)+\alpha\}^2 r_o^2\phi_r + D_{16}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2\phi_\theta}{\partial r^2}$$
$$+ (D_{12}+D_{66})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2\phi_\theta}{\partial r\partial\theta} + D_{26}\frac{1}{\Theta^2}\frac{\partial^2\phi_\theta}{\partial\theta^2} - D_{26}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial\phi_\theta}{\partial r}$$
$$- (D_{22}+D_{66})\frac{1}{\Theta}\frac{\partial\phi_\theta}{\partial\theta} - D_{26}\phi_\theta - k^2A_{45}\{r(1-\alpha)+\alpha\}^2 r_o^2\phi_\theta$$
$$= \{r(1-\alpha)+\alpha\}^2 r_o^2\left(I_2\frac{d^2\phi_r}{dt^2} + I_1\frac{d^2u}{dt^2}\right). \quad \text{(A.4)}$$

- ϕ_θ direction

$$B_{16}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 u}{\partial r^2} + (B_{12}+B_{66})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2 u}{\partial r\partial\theta} + B_{26}\frac{1}{\Theta^2}\frac{\partial^2 u}{\partial\theta^2}$$
$$+ (2B_{16}+B_{26})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial u}{\partial r} + (B_{22}+B_{66})\frac{1}{\Theta}\frac{\partial u}{\partial\theta} - B_{26}u + b_{66}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2 v}{\partial r^2}$$
$$+ 2B_{26}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2 v}{\partial r\partial\theta} + B_{22}\frac{1}{\Theta^2}\frac{\partial^2 v}{\partial\theta^2} - B_{66}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial v}{\partial r}$$
$$- B_{66}v - k^2A_{45}\frac{\{r(1-\alpha)+\alpha\}^2 r_o}{(1-\alpha)}\frac{\partial w}{\partial r} - k^2A_{55}\frac{\{r(1-\alpha)+\alpha\}r_o}{\Theta}\frac{\partial w}{\partial\theta} + D_{16}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2\phi_r}{\partial r^2}$$
$$+ (D_{12}+D_{66})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2\phi_r}{\partial r\partial\theta} + D_{26}\frac{1}{\Theta^2}\frac{\partial^2\phi_r}{\partial\theta^2} + (2D_{16}+D_{26})\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial\phi_r}{\partial r}$$
$$+ (D_{22}+D_{66})\frac{1}{\Theta}\frac{\partial\phi_r}{\partial\theta} - D_{26}\phi_r - k^2A_{45}\{r(1-\alpha)+\alpha\}^2 r_o^2\phi_r$$
$$+ D_{66}\frac{\{r(1-\alpha)+\alpha\}^2}{(1-\alpha)^2}\frac{\partial^2\phi_\theta}{\partial r^2} + 2D_{26}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)\Theta}\frac{\partial^2\phi_\theta}{\partial r\partial\theta} + D_{22}\frac{1}{\Theta^2}\frac{\partial^2\phi_\theta}{\partial\theta^2}$$
$$- D_{66}\frac{\{r(1-\alpha)+\alpha\}}{(1-\alpha)}\frac{\partial\phi_\theta}{\partial r} - D_{66}\phi_\theta - k^2A_{44}\{r(1-\alpha)+\alpha\}^2 r_o^2\phi_\theta$$
$$= \{r(1-\alpha)+\alpha\}^2 r_o^2\left(I_2\frac{d^2\phi_r}{dt^2} + I_1\frac{d^2u}{dt^2}\right). \quad \text{(A.5)}$$

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